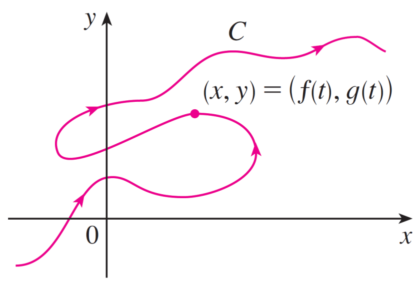
Parametric Curves

Suppose that x and y are both given as functions of a third variable t (called a **parameter**) by the equations

$$x = f(t), \qquad y = g(t)$$

(called **parametric equations**). Each value of t determines a point (x, y), which we can plot in a coordinate plane. As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces out a curve C, which we call a **parametric curve.**



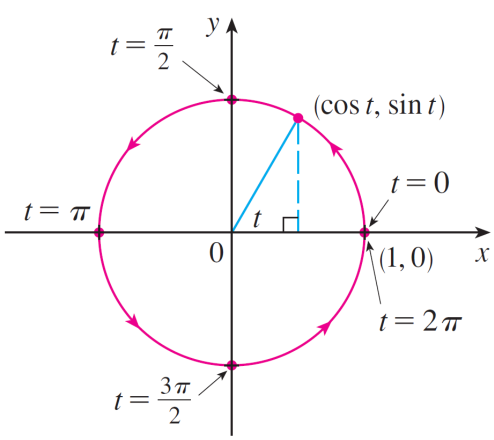
EXAMPLE: Sketch and identify the curve defined by the parametric equations

$$x = \cos t, \quad y = \sin t \quad 0 \leq t \leq 2\pi$$

Solution: If we plot points, it appears that the curve is a circle (see the figure below and page 6). We can confirm this impression by eliminating t. If fact, we have

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1$$

Thus the point (x, y) moves on a unit circle $\boxed{x^2 + y^2 = 1}$.



EXAMPLE: Sketch and identify the curve defined by the parametric equations

$$x = \sin 2t, \quad y = \cos 2t \quad 0 \leq t \leq 2\pi$$

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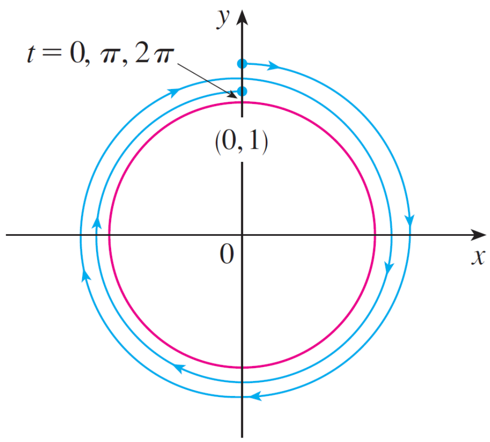
$$x = \sin 2t, \quad y = \cos 2t \quad 0 \le t \le 2\pi$$

Solution: If we plot points, it appears that the curve is a circle (see the Figure below). We can confirm this impression by eliminating t. If fact, we have

$$x^2 + y^2 = \sin^2 2t + \cos^2 2t = 1$$

Thus the point (x, y) moves on a unit circle $\boxed{x^2 + y^2 = 1}$.



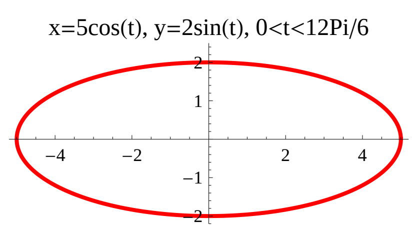
EXAMPLE: Sketch and identify the curve defined by the parametric equations

$$x = 5\cos t, \quad y = 2\sin t \quad 0 \le t \le 2\pi$$

Solution: If we plot points, it appears that the curve is an ellipse (see page 8). We can confirm this impression by eliminating t. If fact, we have

$$\frac{x^2}{25} + \frac{y^2}{4} = \left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$$

Thus the point (x, y) moves on an ellipse $\boxed{\frac{x^2}{25} + \frac{y^2}{4} = 1}$.



EXAMPLE: Sketch and identify the curve defined by the parametric equations

$$x = t\cos t, \quad y = t\sin t \quad t > 0$$

EXAMPLE: Sketch and identify the curve defined by the parametric equations

$$x = t\cos t, \quad y = t\sin t \quad t > 0$$

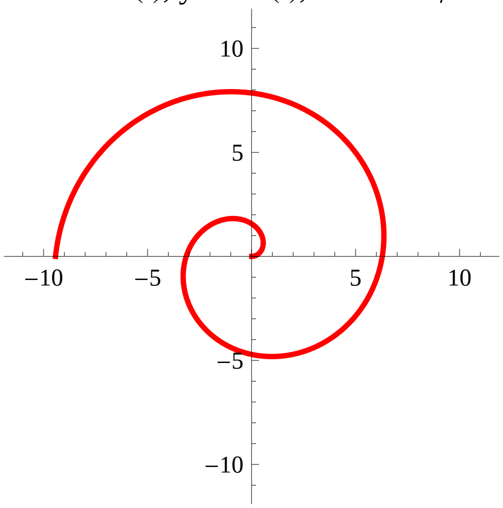
Solution: If we plot points, it appears that the curve is a spiral (see page 9). We can confirm this impression by the following algebraic manipulations:

$$x^2 + y^2 = (t\cos t)^2 + (t\sin t)^2 = t^2\sin^2 t + t^2\cos^2 t = t^2(\sin^2 t + \cos^2 t) = t^2 \quad \Longrightarrow \quad x^2 + y^2 = t^2$$

To eliminate t completely, we observe that

$$\frac{y}{x} = \frac{t\sin t}{t\cos t} = \frac{\sin t}{\cos t} = \tan t \quad \Longrightarrow \quad t = \arctan\left(\frac{y}{x}\right)$$

Substituting this into $x^2 + y^2 = t^2$, we get $\boxed{x^2 + y^2 = \arctan^2\left(\frac{y}{x}\right)}$.



EXAMPLE: Sketch and identify the curve defined by the parametric equations

$$x = t^2 + t, \qquad y = 2t - 1$$

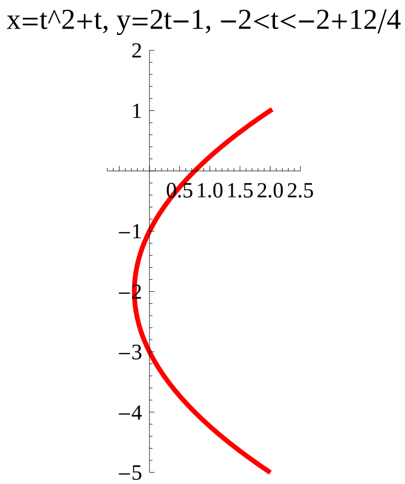
EXAMPLE: Sketch and identify the curve defined by the parametric equations

$$x = t^2 + t, \qquad y = 2t - 1$$

Solution: If we plot points, it appears that the curve is a parabola (see page 10). We can confirm this impression by eliminating t. If fact, we have

$$y = 2t - 1 \quad \Longrightarrow \quad t = \frac{y+1}{2} \quad \Longrightarrow \quad x = t^2 + t = \left(\frac{y+1}{2}\right)^2 + \frac{y+1}{2} = \frac{1}{4}y^2 + y + \frac{3}{4}$$

Thus the point (x, y) moves on a parabola $\boxed{x = \frac{1}{4}y^2 + y + \frac{3}{4}}$.



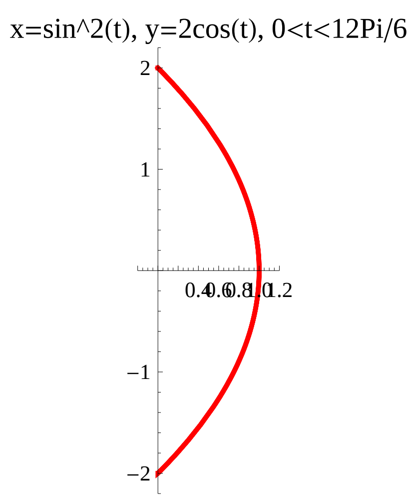
EXAMPLE: Sketch and identify the curve defined by the parametric equations

$$x = \sin^2 t, \qquad y = 2\cos t$$

Solution: If we plot points, it appears that the curve is a restricted parabola (see page 11). We can confirm this impression by eliminating t. If fact, we have

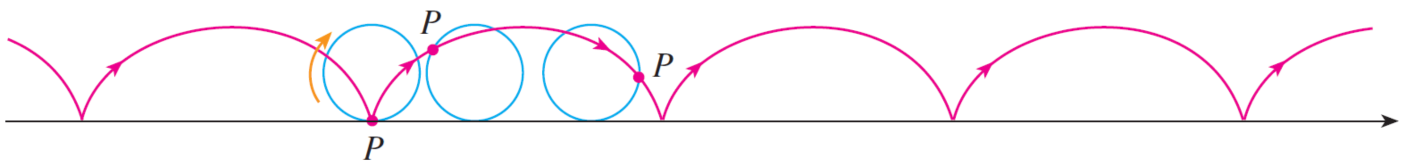
$$y = 2\cos t \quad \Longrightarrow \quad y^2 = 4\cos^2 t \quad \Longrightarrow \quad 4x + y^2 = 4\sin^2 t + 4\cos^2 t = 4 \quad \Longrightarrow \quad x = 1 - \frac{y^2}{4}$$

We also note that $0 \le x \le 1$ and $-2 \le y \le 2$. Thus the point (x, y) moves on the restricted parabola $\boxed{x = 1 - \frac{y^2}{4}}$.



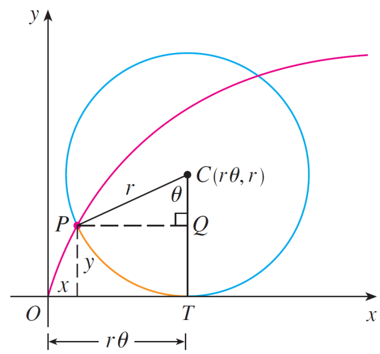
The Cycloid

EXAMPLE: The curve traced out by a point P on the circumference of a circle as the circle rolls along a straight line is called a **cycloid** (see the Figure below). If the circle has radius r and rolls along the x-axis and if one position of P is the origin, find parametric equations for the cycloid.



Solution: We choose as parameter the angle of rotation θ of the circle ($\theta = 0$ when P is at the origin). Suppose the circle has rotated through θ radians. Because the circle has been in contact with the line, we see from the Figure below that the distance it has rolled from the origin is

$$|OT| = \text{ arc } PT = r\theta$$



Therefore, the center of the circle is $C(r\theta, r)$. Let the coordinates of P be (x, y). Then from the Figure above we see that

$$x = |OT| - |PQ| = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = |TC| - |QC| = r - r\cos\theta = r(1 - \cos\theta)$$

Therefore, parametric equations of the cycloid are

$$x = r(\theta - \sin\theta), \quad y = r(1 - \cos\theta) \quad \theta \in \mathbb{R} \tag{1}$$

One arch of the cycloid comes from one rotation of the circle and so is described by $0 \le \theta \le 2\pi$. Although Equations 1 were derived from the Figure above, which illustrates the case where $0 < \theta < \pi/2$, it can be seen that these equations are still valid for other values of θ.

Although it is possible to eliminate the parameter θ from Equations 1, the resulting Cartesian equation in x and y is very complicated and not as convenient to work with as the parametric equations:

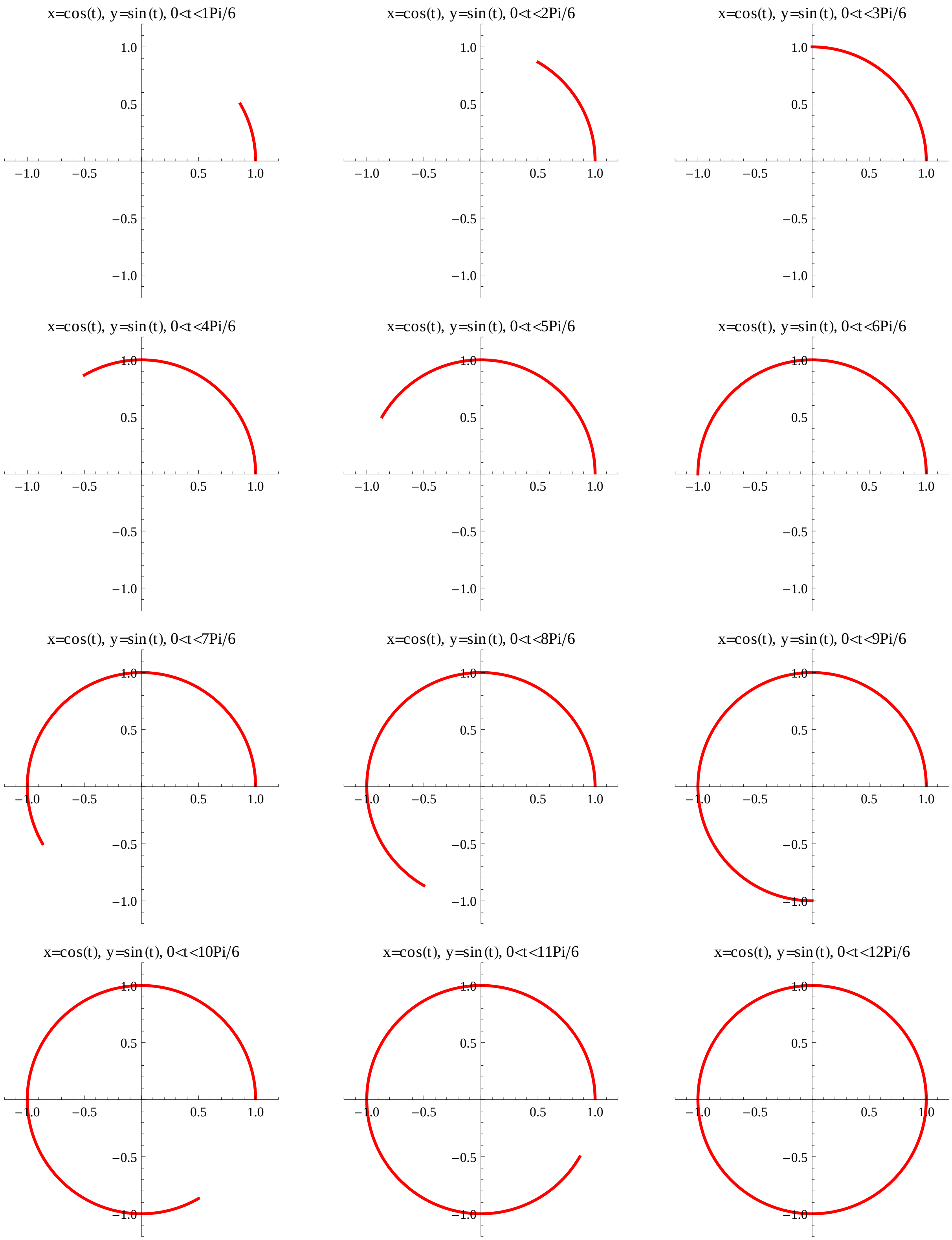
$$\left|\frac{x}{r} + 2\pi\left[\frac{1}{2} - \frac{x}{\pi r}\right] - 1\right| = \cos^{-1}\left(1 - \frac{y}{r}\right) - 2\sqrt{2\frac{y}{r} - \left(\frac{y}{r}\right)^2}$$

Unit Circle

Parametric equations:

$$x = \cos t, \quad y = \sin t$$

where $t = k\pi/6,\ k = 0, \dots, 12.$

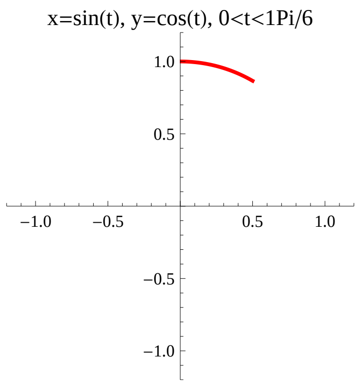


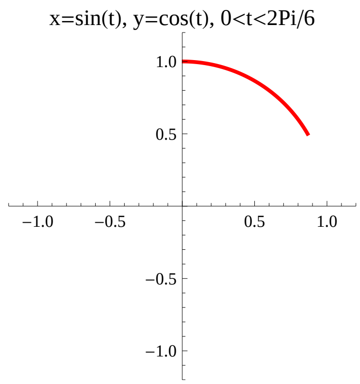
Unit Circle

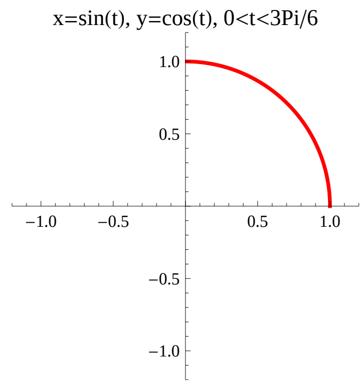
Parametric equations:

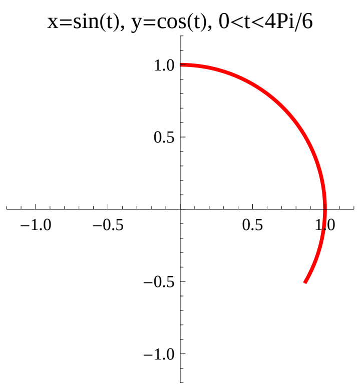
$$x = \sin t, \quad y = \cos t$$

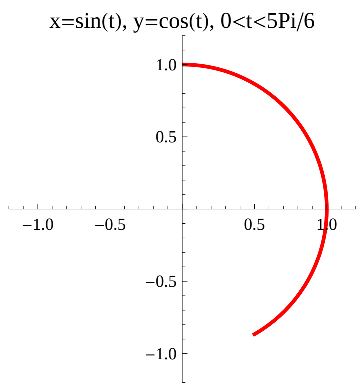
where $t = k\pi/6$, $k = 0, \dots, 12$.

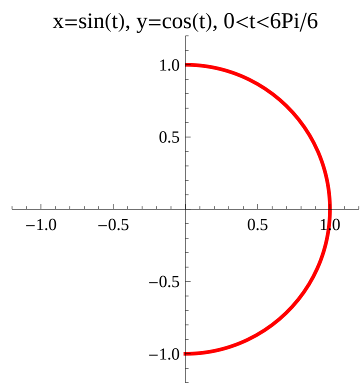


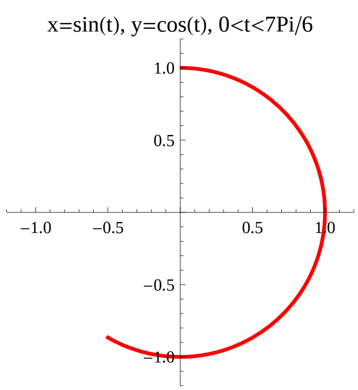


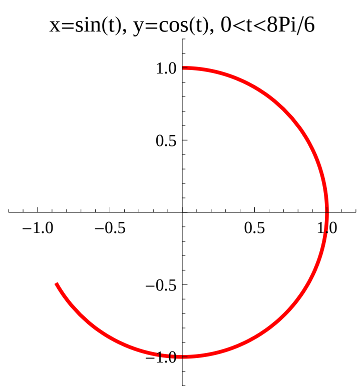


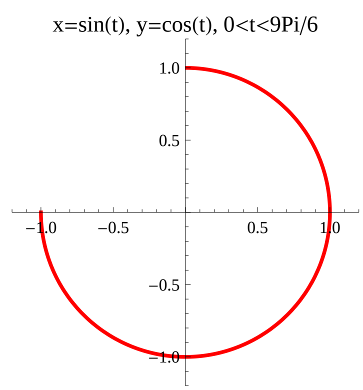


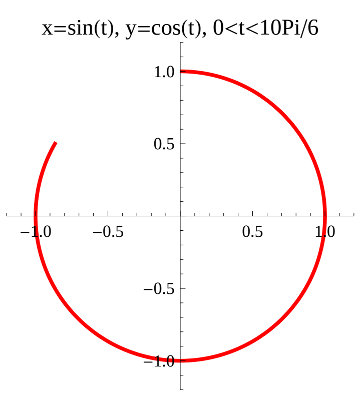


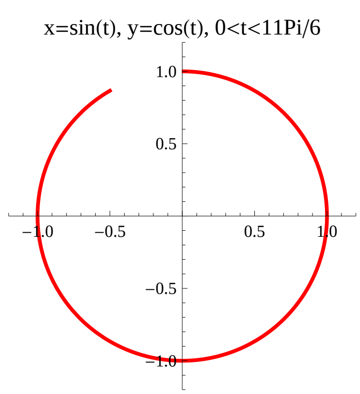


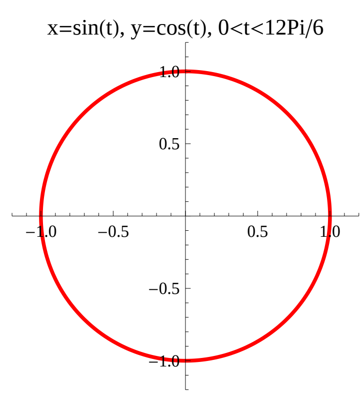










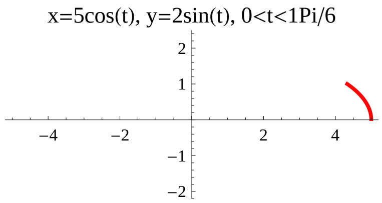


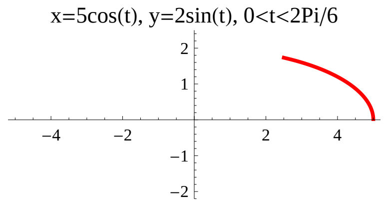
Ellipse

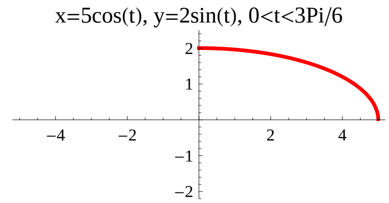
Parametric equations:

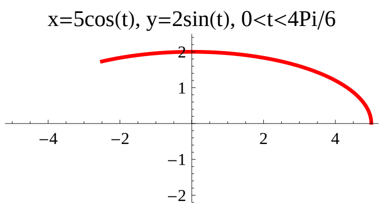
$$x = 5\cos t, \quad y = 2\sin t$$

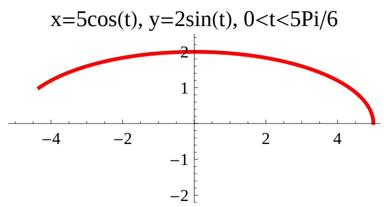
where $t = k\pi/6,\ k = 0, \dots, 12.$

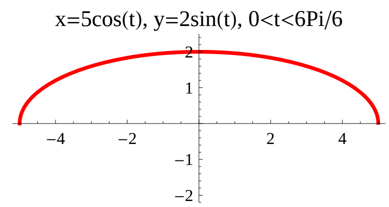


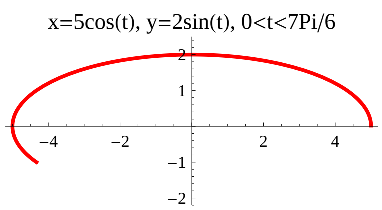


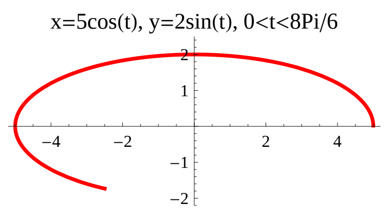


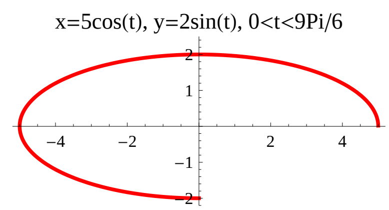


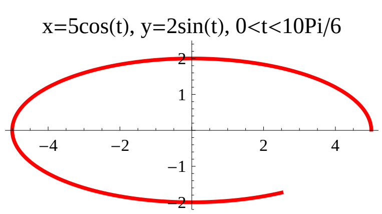


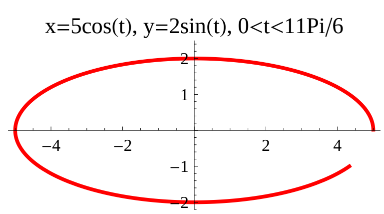


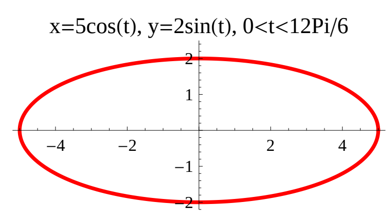










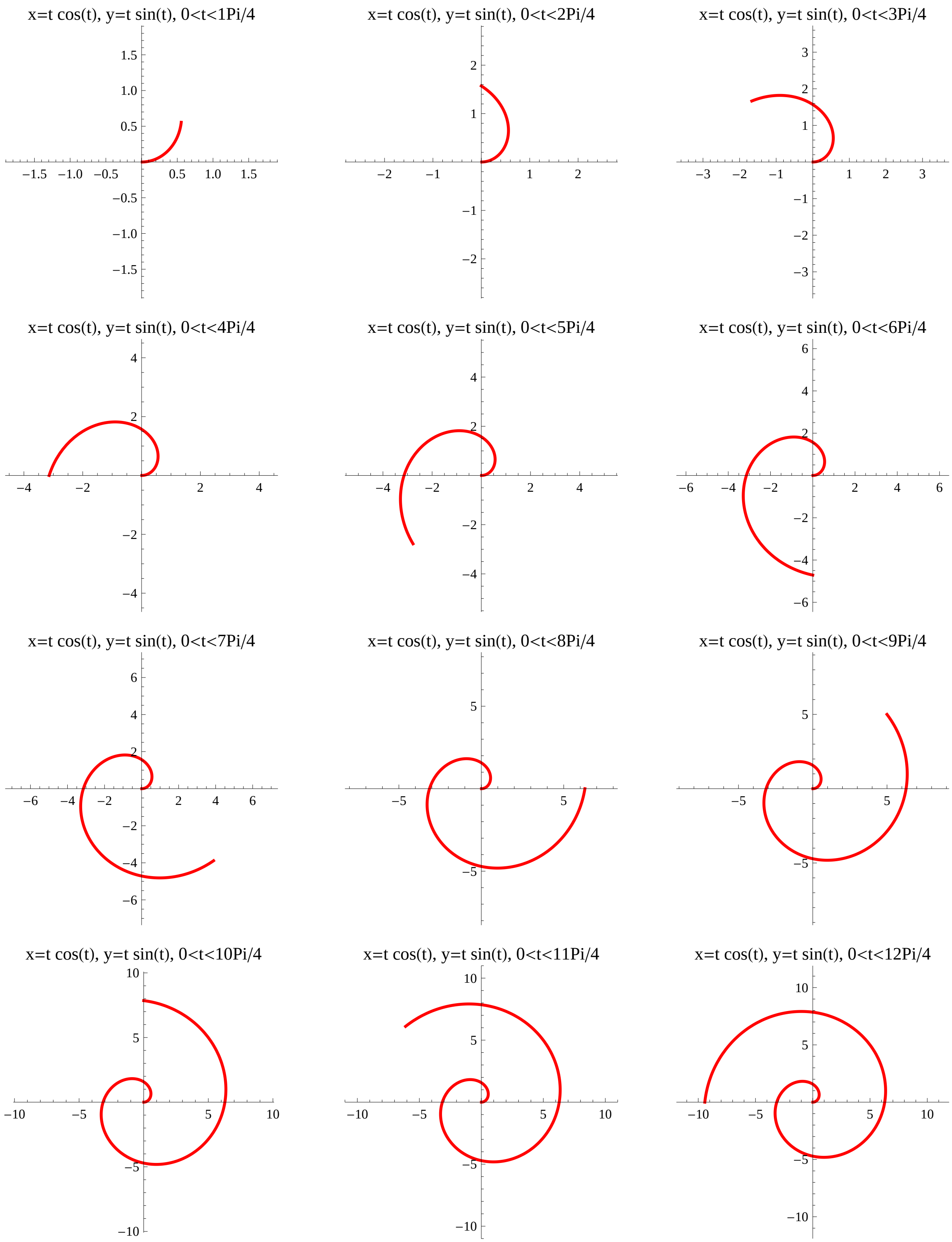


Spiral

Parametric equations:

$$x = t\cos t, \quad y = t\sin t$$

where $t = k\pi/4,\ k = 0, \ldots, 12.$

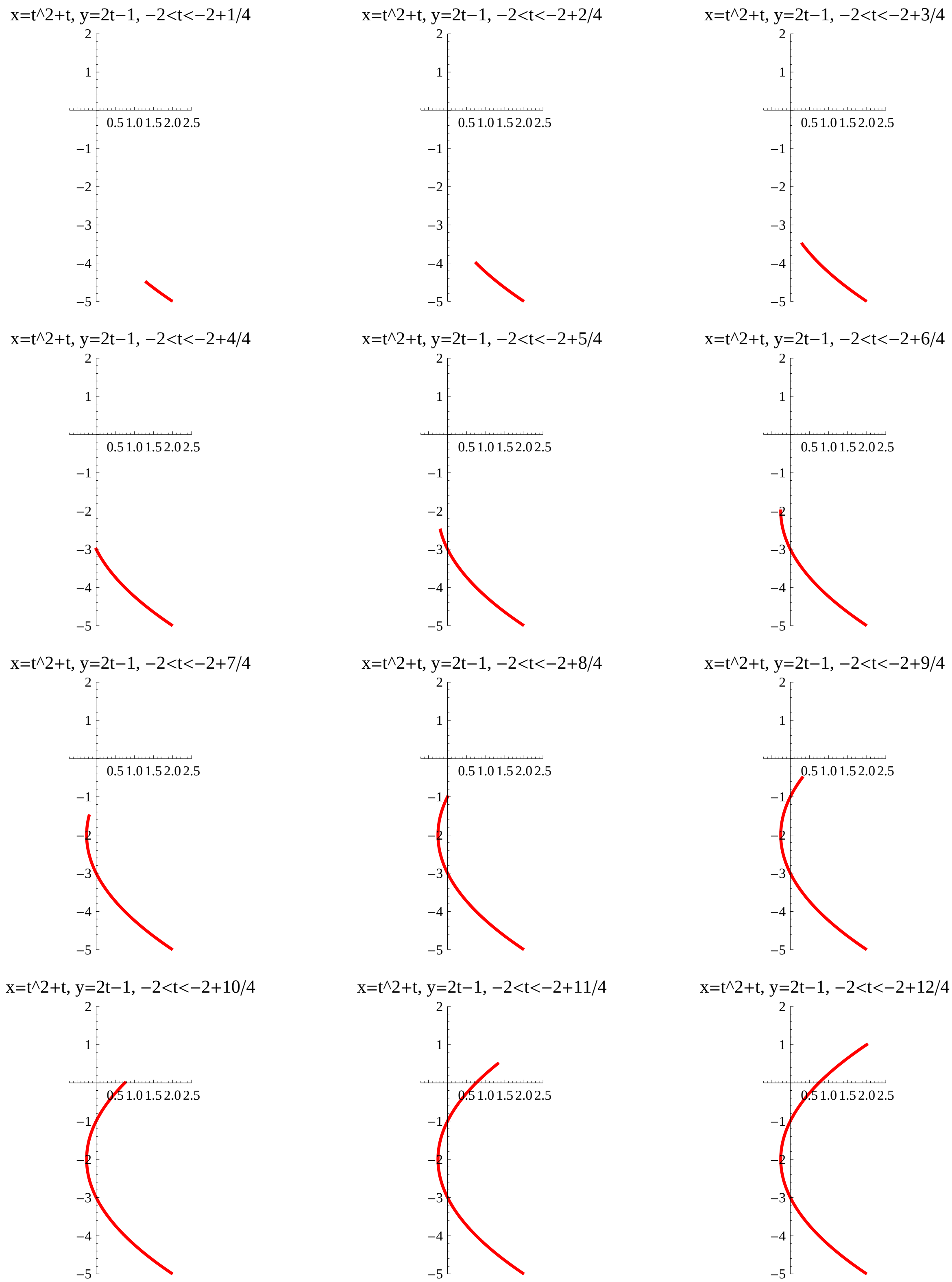


Parabola

Parametric equations:

$$x = t^2 + t, \quad y = 2t - 1$$

where $t = -2 + k/4,\ k = 0, \dots, 12.$

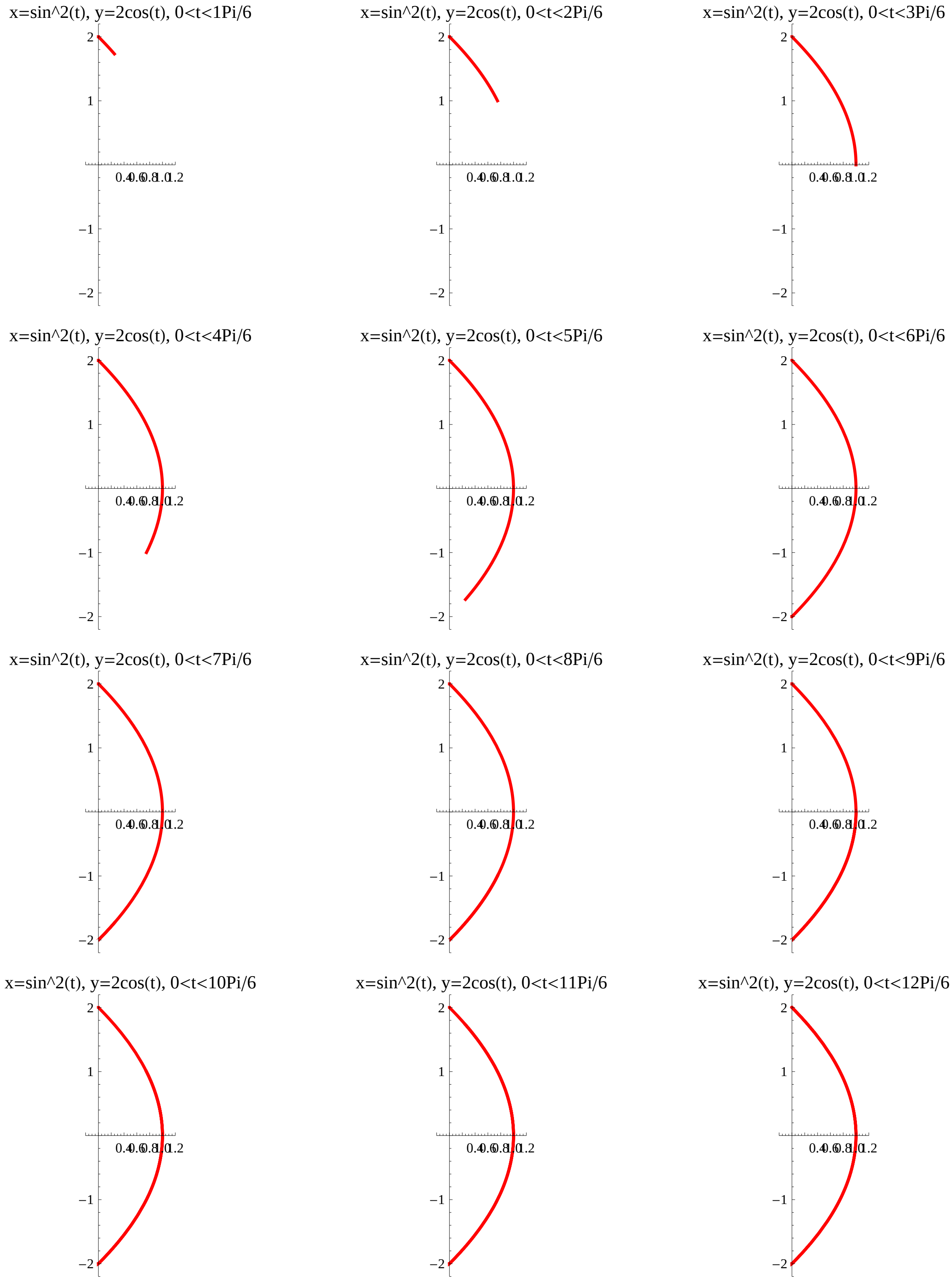


Restricted Parabola

Parametric equations:

$$x = \sin^2 t, \quad y = 2\cos t$$

where $t = k\pi/6,\ k = 0, \ldots, 12.$

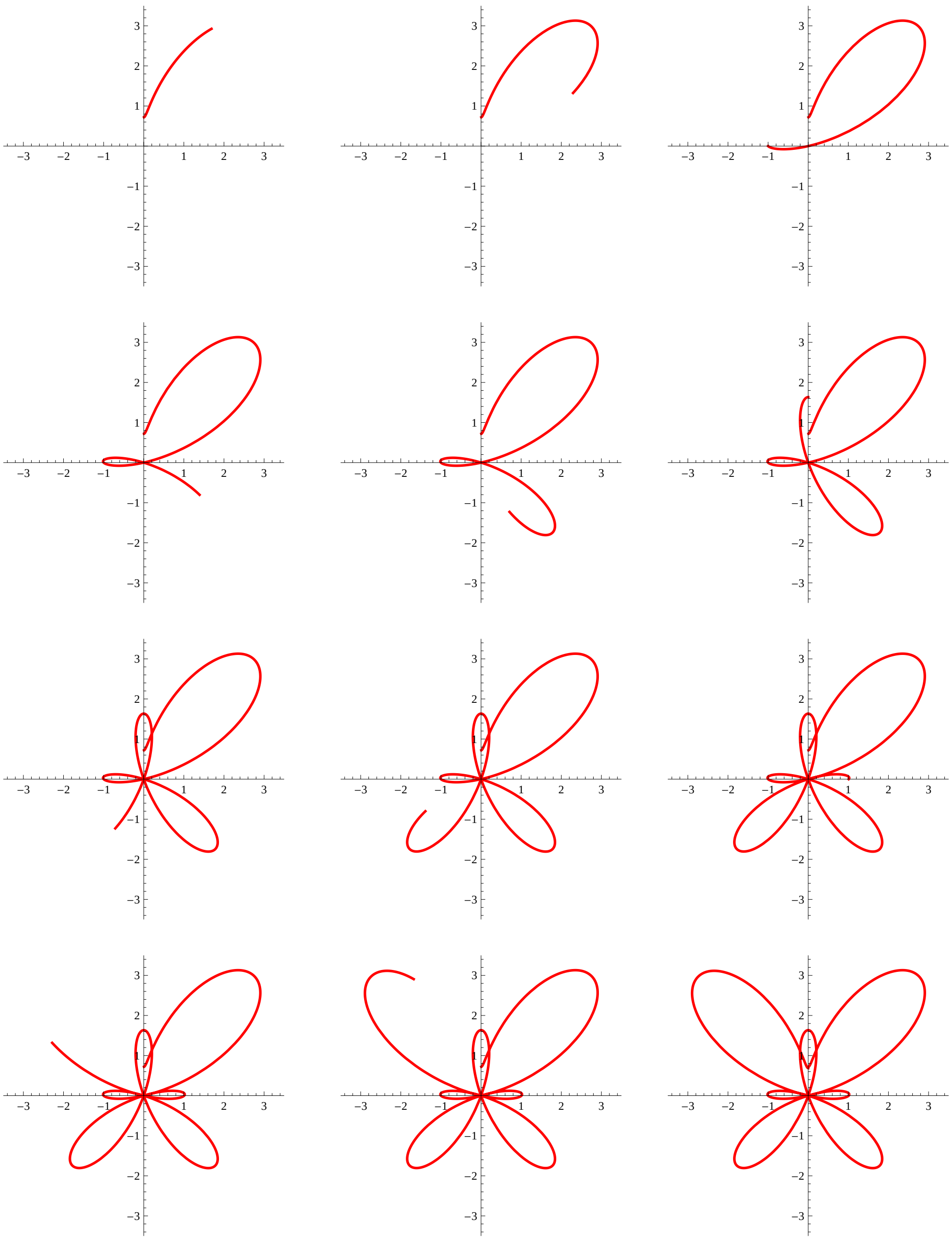


Butterfly Curve

Parametric equations:

$$x = \sin t \left[e^{\cos t} - 2\cos(4t) + \sin^5\left(\frac{t}{12}\right)\right], \qquad y = \cos t \left[e^{\cos t} - 2\cos(4t) + \sin^5\left(\frac{t}{12}\right)\right]$$

where $t = k\pi/6,\ k = 0, \ldots, 12$.

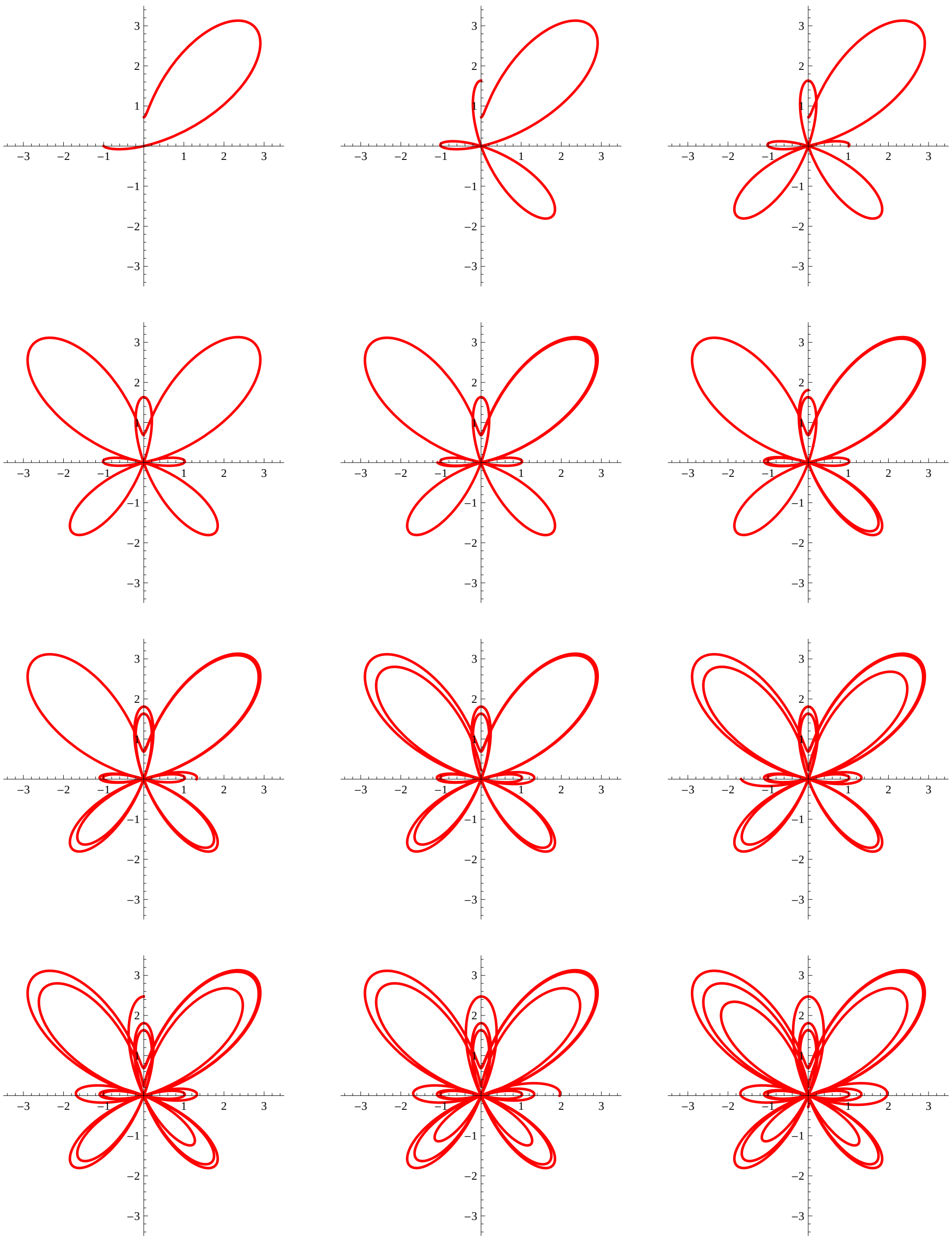


Butterfly Curve

Parametric equations:

$$x = \sin t \left[e^{\cos t} - 2\cos(4t) + \sin^5\left(\frac{t}{12}\right) \right], \qquad y = \cos t \left[e^{\cos t} - 2\cos(4t) + \sin^5\left(\frac{t}{12}\right) \right]$$

where $t = k\pi/2,\ k = 0, \ldots, 12.$

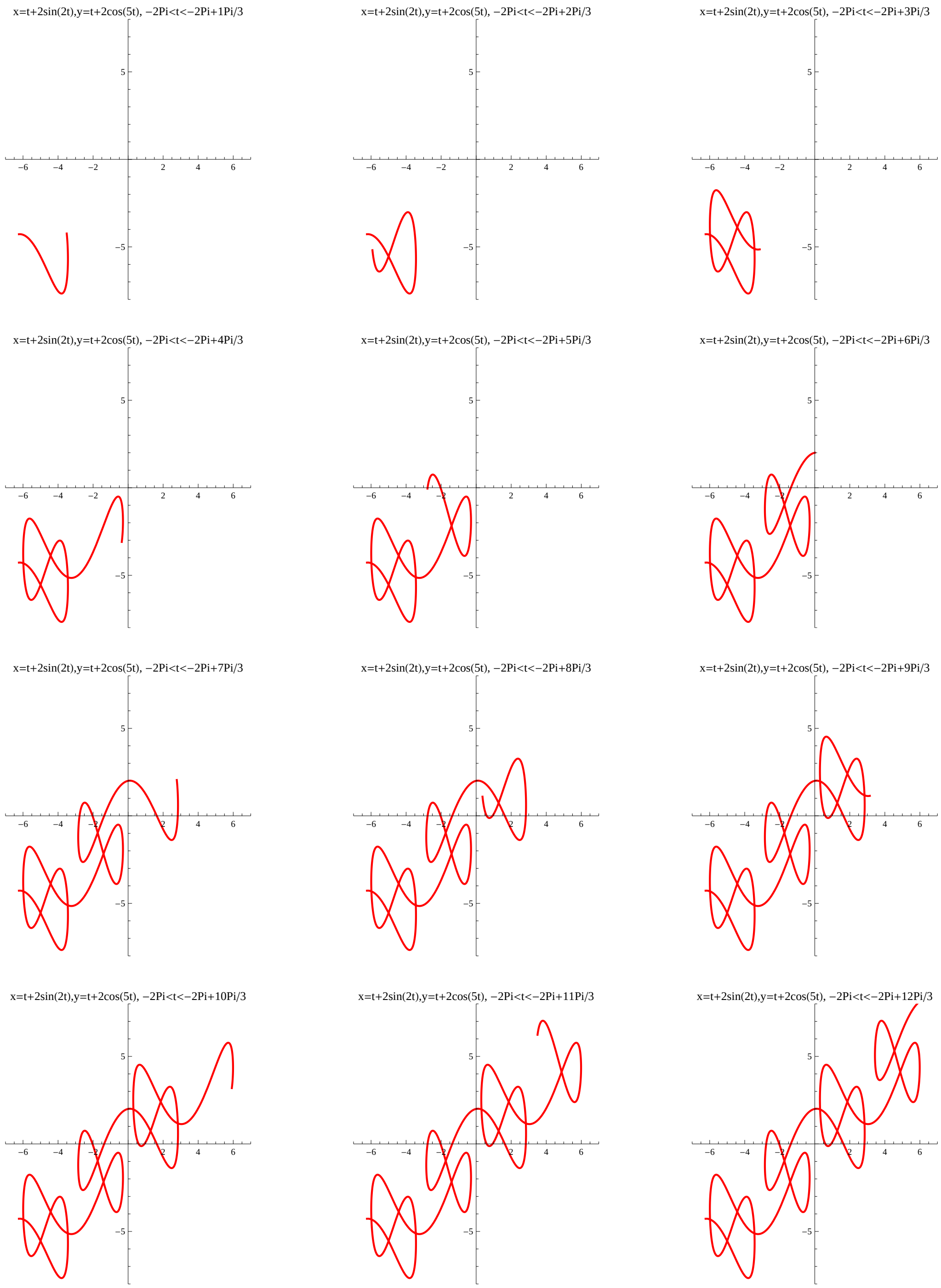


Parametric Curve

Parametric equations:

$$x = t + 2\sin 2t, \quad y = t + 2\cos 5t$$

where $t = -2\pi + k\pi/3,\ k = 0, \ldots, 12.$

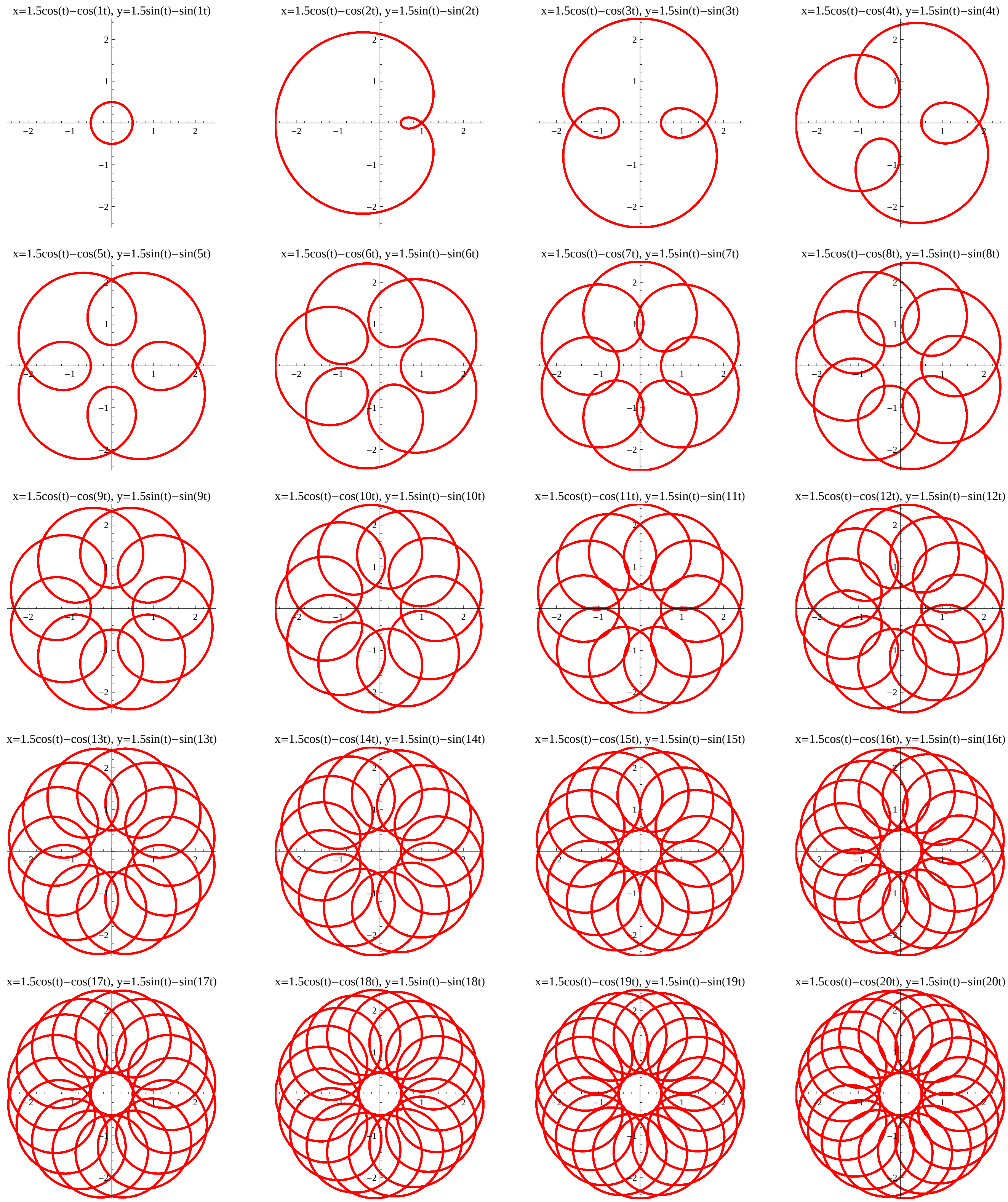


Parametric Curves

Parametric equations:

$$x = 1.5\cos t - \cos kt, \quad y = 1.5\sin t - \sin kt$$

where $k = 1, \ldots, 20$.



Parametric Curves

Parametric equations:

$$x = 1.5\cos t - \cos 5t, \quad y = 1.5\sin t - \sin kt$$

where $k = 1, \ldots, 20$.